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ON THE MOTION OF FLUID NEAR A STRETCHING CIRCULAR CYLINDER*

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New exact solutions of the Navier-Stokes equations are obtained, describing the stationary axisymmetric motions of an incompressible fluid near an infinite circular cylinder whose surface stretches and expands in the axial direction, so that the axial velocity component at the boundary is linear in the corresonding coordinate.

The solutions for the case of a stretching plane were discussed in /1-3/. In /4/, the solution of /1/ was interpreted as the motion of a fluid with a free surface, caused by a tangential force applied to the surface.

1. Let us consider the axisymmetric stationary motion of a viscous incompressible fluid about an infinite circular cylinder of radius R, stretching along its axis. The axial and radial component of the velocity, u and v, are given at the cylinder surface in cylindrical x, r coordinates (the x axis is directed along the axis of the cylinder) by the relations

$$r = R, \quad u = kx, \quad v = 0 \tag{1.1}$$

Seeking the solutions of the Navier-Stokes equations in the form

$$u = \sqrt{\nu k}\left[g(\xi) + \sqrt{\frac{k}{\nu}}\,\varphi'(\xi)\,x\right], \qquad v = -\sqrt{\nu k}\,\frac{\varphi(\xi)}{\sqrt{2\xi}} \tag{1.2}$$

$$p = -\rho\nu k\left[\frac{kc^2}{2\nu}x^2 + \sqrt{\frac{k}{\nu}}\,ax + F(\xi)\right]; \qquad \xi = \frac{kr^2}{2\nu}$$

where a and c are constants, we arrive at the following system of equations for the functions φ, g, F:

$$\begin{gathered} 2(\xi\varphi'')' + \varphi\varphi'' - \varphi'^2 + c^2 = 0 \\ 2(\xi g')' + g'\varphi - g\varphi' + a = 0 \\ 4\xi^2 F' = 4\xi^2\varphi'' + 2\xi\varphi\varphi' - \varphi^2 \end{gathered} \tag{1.3}$$

The boundary conditions at the cylinder surface follow from (1.1)

$$\xi = \xi_w, \quad \varphi = 0, \quad \varphi' = 1, \quad g = 0; \quad \xi_w = kR^2/(2\nu) \tag{1.4}$$

The form of the boundary conditions at infinity is determined by the presence and structure of the external flow.

2. We begin our investigation of the solutions of system (1.3) by considering the case when the motion of the fluid occurs only as a result of stretching of the cylinder (there is no external flow). We have for this case $c = 0, g = 0, a = 0$ and a boundary condition of the form $\xi \to \infty, \varphi' = 0$.

A solution of the equation for φ satisfying this condition and the first condition of (1.4), has the form

$$\varphi = 8(\xi - \xi_w)/(2\xi + \xi_w) \tag{2.1}$$

The second condition of (1.4) will be satisfied only when a definite relation connects the parameters k, ν, R:

$$k = 16\nu/(3R^2) \tag{2.2}$$

Determining the function F from the last equation of (1.3) and using (1.2), we arrive at the relations for the velocity and pressure components, which yield a solution of the problem under the condition (2.2)

$$u = \frac{9kx}{\alpha^2}, \qquad v = -\sqrt{12\nu k}\,\frac{\eta - 1}{\sqrt{\eta}\,\alpha} \tag{2.3}$$

$$p = -3\rho\nu k\,\frac{2\eta^2 - \eta + 2}{\eta\alpha^2}; \qquad \eta = \frac{r^2}{R^2}, \qquad \alpha = 2\eta + 1$$

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In the case when the stretching cylinder is streamlined by a homogeneous flow $(r\to\infty,\ u=U_\infty)$ parallel to the x axis, we find the function g from the second equation of (1.3) and obtain, at $a=0$, taking into account (1.4), (2.1) and (2.2),

$$u=9k\alpha^{-2}x+2\alpha^{-2}U_\infty\left[2\eta^2-4\eta+\alpha^{-1}+3\ln\alpha+A\right];\ A={}^5/_3-3\ln 3 \tag{2.4}$$

In the case when the external flow is a shear flow $(r\to\infty,\ u=A_1\xi+A_2\ln\xi+A_3)$, the velocity distribution is found from the solution of Eqs.(1.3) at $a\neq 0$. We omit the corresponding formulas for brevity.

The solutions (2.1)-(2.4) are analogous to the solutions obtained in /1, 2/ for the case of a stretching plate. The solution of /1/, as well as (2.2) and (2.3), describe a flow appearing as a result of stretching of the surface, and the uniqueness of the solution given in /1/ was proved in /3/. The solution given in /2/, as well as (2.2) and (2.4), describe a homogeneous flow past a stretching surface.

Let us compare (2.2) and (2.3) with the solution given in /1/, in the region of flow near the surface $(r-R\ll R)$. The solution given in /1/ has the form

$$v_x=kxe^{-\beta},\quad v_y=-\sqrt{k\nu}\,(1-e^{-\beta}),\quad \beta=\sqrt{k/\nu}\,y$$

(the y axis is perpendicular to the plane) and at small β it is given by the expressions

$$v_x=kx\,(1-\beta),\quad v_y=-\sqrt{k\nu}\beta \tag{2.5}$$

We put $r=R+y$ in (2.3) and use (2.2) to obtain, for $y\ll R$,

$$\eta=(1+\sqrt{3}\beta/4)^2,\quad \beta=\sqrt{k/\nu}\,y$$
$$u=kx\,(1-2\beta/\sqrt{3}),\quad v=-\sqrt{k\nu}\beta$$

Comparing the above expressions with (2.5) we see, that the influence of the curvature of the surface on the form of the solution manifests itself even at distances small compared with the radius of the cylinder.

Let us now consider the motion of the fluid near the stretching cylinder under the conditions that the fluid impinges on the cylinder at infinity in the radial direction, and spreads in opposite directions from the circle $x=0$:

$$r\to\infty,\quad u=kcx,\quad v=-{}^1/_2kcr$$

In this case the first equation of (1.3) must be solved at $c\neq 0$, with boundary condition at infinity of the form

$$\xi\to\infty,\quad \varphi'=c$$

In this case we can also find an exact solution of the equations for φ, satisfying the condition at infinity and one of the conditions of (1.4)

$$\varphi=c\xi-6-(c\xi_w-6)\,e^{c(\xi_w-\xi)/2} \tag{2.6}$$

The second condition of (1.4) will be satisfied provided that the following relation holds:

$$k=4\nu\lambda R^{-2}c^{-1},\quad \lambda=2+c^{-1} \tag{2.7}$$

We will write expressions for u and v, corresponding to (2.6) and (2.7), and expressions for the pressure obtained by determining the function F from the last equation of (1.3)

$$\begin{aligned}&u=kcx\,(1+f),\quad f=(\lambda-3)\,e^{\lambda(1-\eta)}\\&v=-{}^1/_2kcr\,(\lambda\eta)^{-1}(\lambda\eta-3-f)\\&p=-{}^1/_2\rho\nu kc\,[kc\nu^{-1}x^2+\lambda\eta+(3+f)^2\lambda^{-1}\eta^{-1}]\end{aligned} \tag{2.8}$$

In the special case of $c=1$, the solution of the first equation of (1.3) has the form $\varphi=\xi-\xi_w$ (this solution satisfies the conditions (1.4) without restricting (2.7)). The corresponding velocity distribution differs from the potential flow only in the term $\sim 1/r$ in the expression for v. The solution at $c=1$ admits of a fairly simple generalization to the case when the plane of radial flow of the fluid at infinity impinging on the cylinder is displaced relative to the fixed circle $x=0$:

$$r\to\infty,\ u=k\,(x+b),\ v=-{}^1/_2kr$$

Determining g from the second equation of (1.3) at $a=\sqrt{k/\nu}\,b$ and taking into account (1.2), we obtain

$$\begin{aligned}&u=k\,(x+b)+A\,(\xi+m)\int\frac{\xi^{-m/2}e^{-\xi/2}}{(\xi+m)^2}\,d\xi\\&v=-{}^1/_2kr\,(\xi-\xi_w)/\xi,\quad m=2-\xi_w\end{aligned} \tag{2.9}$$

The constant A is found from condition (1.1), and we assume the integration constant to be equal to zero. We can carry out the integration in (2.9) for discrete values of $m = -2n\ (n = 0, 1, 2, \ldots)$. Let us write out the expressions obtained for the first three values of m

$$u = k\{x + b[1 - H(m, \eta)]\}$$
$$H(0, \eta) = [\eta\,\mathrm{Ei}(-\eta) + e^{-\eta}]/[\mathrm{Ei}(-1) + e^{-1}]$$
$$H(-2, \eta) = e^{2(1-\eta)}, \quad H(-4, \eta) = {}^1/_5\,(3\eta + 2)\,e^{3(1-\eta)}$$

where $\mathrm{Ei}(z)$ is an integral exponential function.

The solutions discussed above can also be used in a situation when the surface of the cylinder not only stretches, but also moves with constant velocity in the direction of the x axis. In this case we replace the boundary conditions (1.1) by the relations

$$r = R, \quad u = kx + U_w, \quad v = 0 \tag{2.10}$$

Solved (2.4) satisfies this condition at another value of the constant A:

$$A = {}^5/_3 - 3\ln 3 - {}^9/_2 U_w/U_\infty$$

The solution (2.6)-(2.8) can be generalized to the case (2.10), provided that we add the term $\rho k U_w c^2 x$, to p in (2.8) and the term $\sqrt{\nu k}g$ to u, where

$$g = U_w c\,(\nu k)^{-1/2}\,[1 + (\lambda - 3)\,e^{\lambda(1-\eta)}]$$

The solution (2.9) can be generalized to the case (2.10) by leaving the constant A undetermined.

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ON THE PROBLEM OF THE COLLAPSE OF CAVITATIONAL VOIDS*

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The part played by the capillary properties of a medium in the problem of the collapse on an empty spherical cavity in a viscous incompressible fluid modelling the stage of collapse of cavitational voids is studied. Methods of qualitative theory are used to study the differential equations describing the dynamics of the boundary of the cavity. A pattern of behaviour of the integral curves in the phase plane is obtained and used to produce a complete description of all possible modes of collapse of the cavity.

The problem of the filling of an empty spherical cavity with an ideal incompressible fluid was studied by Rayleigh /1/, who showed that the velocity of the liquid boundary of the cavity increases without limit as $R^{-3/2}$ as its radius R decreases to zero. The time in which the cavity disappears is always finite.

Taking into account the viscosity of the fluid /2/ leads to the conclusion that a critical Reynolds number Re^* exists, separating two, essentially different modes of filling the cavity. When $\mathrm{Re} > \mathrm{Re}^*$, the character of the motion is analogous to that in Rayleigh's case. The principal term of the expansion of the velocity V of the boundary of the cavity

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